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ENCOUNTERS OF EXACT DIFFERENTIAL ON $udx + vdy + wdz$ WITH VORTICITY AND THE NAVIER-STOKES EQUATIONS. - 150 YEARS SINCE HELMHOLTZ'S PAPER

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ABSTRACT. In the fluid mechanics, it is an important concept to analyze it, for example in three variables, for $udx+vdy+wdz$ to be satisfied with an *exact differentiability* or a *complete differentiability*. By d'Alembert, Euler, Lagrange, Laplace, Cauchy, Poisson and Stokes succeeded its theoretical side. From the geometrical point of view, Gauss and Riemann applied it. Moreover Helmholtz and W.Thomson applied it to the theory of vorticity. To Helmholtz's vorticity equation, Bertrand criticized but Saint-Venant sided with him. We would like to report it from the historical view of fluid mechanics according to Table 1. On the other hand, the formulations of the two-constant theory in the equilibrium/motion and of the Navier-Stokes equations in the motion was deduced, which show in Table 2, 3.

1. From the observation of exact differential to vortex

1.1. Introduction - the mathematical historic view of the exact differential.

There are theories of equiliblium, applications and discussions about an exact differentiability of $udx + vdy + wdz$ for the fluid mechanics. We show such a historical view in Table 1, in which the topics are : condition of equilibrium of fluid, proof of the eternal continuity in time and space of an exact differential, curvature, electro magnetic, topology, vorticity, discussion on Helmholtz's papers, other applications.

Our motivation to study these theme is due to the article 73, the last pages of Poisson [33, pp.173-4], in which he remarks that because the exact differential holds water in original time of motion, it doesn't follow that it always holds water in all other time interval.[1] We would like to tell the mistake of "*Poisson's conjecture*" and the fact that Navier-Stokes equations are formulated in this stream through these topics.

1.2. Clairaut's *effort* and *exact diffirential*.

Clairaut uses already *effort* (response) and *exact diffirential* on the hydrostatics in 1740 . In his thesis : *Théorie de la figure de la terre, tirée des principes de l'hydrostatique* (Theory of the figure of the earth, come from the principle of the hydrostatics), he proposes *exact diffirential* earlier than Euler.

> Si on voulait présentement faire usage de cette quantité, pour trouver en termes finis la valeur du poids du canal ON, en supposant que la courbure de ce canal fût donnée par une équation entre x et y, on commencerait par faire évanouir y et dy de $Pdy + Qdx$; cette différentielle n'ayant plus que des x et dx, on intégrait en observant de compléter l'intégle, c'est-à-dire d'ajouter la constante nécessaire, afin que le poids fût nul, lorsque x serait égal à CG ; on ferait ensuite $X = CI$, et l'on aurait le poids total de ON. Mais comme l'equilibre du fluide demande que le poids de ON ne dépende pas de la courbure de OSN, c'est-à-dire de la valeur particulière de y en x, il faut donc que $Pdy + Qdx$ puisse s'intégrer sans connaître la valeur de x, c'est-à-dire qu'il faut que $Pdy + Qdx$ soit une *différentielle complète*, afin qu'il puisse y avoir équilibre dans le fluide. [6, §16, p.35-37].

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[1]Mais la démonstration qu'on donne de cette proposition suppose que les valeurs de u, v, w, doivent satisfaire non seulment aux équations différentielles du mouvement, mais encore à toutes celles qui s'en déduisent en les différentiant par rapport à t; ce qui n'a pas toujours lieu à l'egard des expressions de u, v, w, en séries d'exponentielles et de sinus ou cosinus dont les posans et les arcs sont propotionnels au temps; et la demonstration étant alors en défaut, il peut arriver que la formule $udx + vdy + wdz$ soit une différentielle exacte à l'origine du mouvement, et qu'elle ne soit plus à toute autre époque. [33, p.174]

TABLE 1. Theories, applications and discussions about an exact differentiability of $udx+vdy+wdz$ for the fluid mechanics

equilibrium	proof	curvature	electro magnatic	topology	vorticity	discussion	application
Maupertuis 1740,68 [27, 28]							
Clairaut 1741-43,1808(2ed.) [6]					Clairaut 1741-43,1808(2ed.) [6]		
d'Alembert 1749-52 [8]							
Euler 1752-55 [13]					Euler[E226] 1755-57 [13]		
d'Alembert 1761 [9]					d'Alembert 1761 [9]		
Lagrange 1781-1869 [23]	Lagrange 1781-1869 [23]				Lagrange 1781-1869 [23]		
Laplace 1806/07-29 [25]							
Cauchy 1815-27 [5]	Cauchy 1815-27 [5]						
Navier 1822-27 [30]							
Gauss 1813 [14], 1830 [16]		Gauss 1828 [15]					
Poisson 1829-31 [33]							
	Power 1842-42 [35]						
Stokes 1845-49 [39]	Stokes 1845-49 [39]				Stokes 1845-49 [39]		
			Green 1850 [17]				
				Riemann 1857 [36]			
					Helmholtz 1858 [18]	Helmholtz 1868 [19, 20, 21]	
							Clebsch 1858-1859 [7]
	Thomson 1867-69 [43]				Thomson 1867-69 [43]	Thomson 1867-69 [43]	
						Bertrand 1868 [1, 2, 3, 4]	
						Saint-Venant 1868 [38]	
	Lamb 1879 [34]						
							Leray 1934 [26]

Clairaut comments the exact differential[2] in his footnote as follows :

[2]It is called of the condition of the *exact differential* as follows. Now in brief, we treat only two variables.
In the domain K of the plane xy, when the two function $\varphi(x,y) \in C^1$ and $\psi(x,y) \in C^1$ are given, and we suppose

$$\varphi(x,y)dx + \psi(x,y)dy \tag{1}$$

is the total differential of an arbitrary function $F(x,y)$, namely $dF = \varphi dx + \psi dy$. Hence, $F_x = \varphi$, $F_y = \psi$.
Then by the assumptiom, we get $F_{xy} = \varphi_y$ and $F_{yx} = \psi_x$, namely,

$$\varphi_y = \psi_x. \tag{2}$$

J'entends par *différentielle complète*, une quantité qui a pour intégrale une function de x et de y. $ydx + xdy$, $\frac{ydx+xdy}{2\sqrt{a^2+xy}}$ sont des *différentielle complètes*, parce qu'elles ont pour intégrales xy, $\sqrt{a^2+xy}$, $\frac{xdy-ydx}{x^2+y^2}$ est aussi une *différentielle complète*, parce que son intégrale est représentée par l'arc dont la tangente est $\frac{y}{x}$, le rayon étant 1. Mais $y^3dx + x^3dy$, $y^2dx + x^2dy$, ne sont pas des *différentielles complètes*, parce qu'aucunes fonctions de x et de y n'en sauraient être les intégrales. [6, p.37, footnote]. [3]

1.3. Euler's study on the exact differential.

He investigates the characters of the exact differential in the following papers.

- (E258) *Principia motus fluidorum* (Principles of the motion of fluids) [1752], (1756/57), 1761.
- (E225) *Principes généraux de l'état d'équilibre des fluides* [1753], (1755), 1757.
- (E226) *Principes généraux du mouvement des fluides* [1755], (1755), 1757.
- (E227) *Continuation des recherches sur théorie du mouvement des fluides* [1755], (1755), 1757.
- (E375) *Sectio prima de statu aequiliblii fluidorum* (Section 1. On the state of equilibrium of fluids) (1768), 1769.
- (E396) *Sectio secunda de principiis motus fluidorum* (Section 2. On the principles of motion of fluids) (1769), 1770.

where (E...) shows the *Eneström Index* and the following years are :

- year in [], commented by C.Truesdell [44],
- year in (), commented by *Eneström* in *Euleri Opera Omnia* [13],
- published years commented by *Eneström* in *Euleri Opera Omnia* [13],

respectively.

1.3.1. Development of the exact differential by Euler.

Euler investigate the exact differential in many parts, We show one of them as follows. In (E396), Euler questions Problem 34 :

> §88. Si cuiusque fluidi elementi ternae celeritates u, v, w ita sint compatoratae, ut formula $udx+vdy+wdz$ integrationem admittat, aequationem, qua pressio fluidi exprimitur, evolvere. (E396) [13, p.127].
>
> (Translation) $\Rightarrow$ If the three elements of the velocity of an arbitrary fluid : u, v and w are equal to each other and the formula : $udx+vdy+wdz$ is integrable, then to extract the equation, in which the pressure of fluid is expressed.

Euler solves his problem as follows : $dI = udx+vdy+wdz+\Phi dt$, $U = u\left(\frac{du}{dx}\right)+v\left(\frac{du}{dy}\right)+w\left(\frac{du}{dz}\right)+\left(\frac{du}{dt}\right)$. $\frac{du}{dy} = \frac{dv}{dx}$, $\frac{du}{dz} = \frac{dw}{dx}$, $\frac{du}{dt} = \frac{d\Phi}{dx}$. Namely : $U = u\left(\frac{du}{dx}\right) + v\left(\frac{dv}{dx}\right) + w\left(\frac{dw}{dx}\right) + \left(\frac{d\Phi}{dx}\right)$, $V = u\left(\frac{du}{dy}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dw}{dy}\right) + \left(\frac{d\Phi}{dy}\right)$, $W = u\left(\frac{du}{dz}\right) + v\left(\frac{dv}{dz}\right) + w\left(\frac{dw}{dz}\right) + \left(\frac{d\Phi}{dz}\right)$. And now, we postulate that the outer forces P, Q, R act such that : $\int(Pdx + Qdy + Rdz) = S$, and in the fluid element, we consider the pressure $= p$ and the density $= q$, then $\frac{2gdp}{q} = 2gdS - Udx - Vdy - Wdz$, in which we assume the time t is constant, $dx\left(\frac{d\Phi}{dx}\right) + dy\left(\frac{d\Phi}{dy}\right) + dz\left(\frac{d\Phi}{dz}\right) = d\Phi$, $Udx + Vdy + Wdz = udu + vdv + wdw + d\Phi$. When this formula is integlable, then $2g\int\frac{dp}{q} = 2gS - \frac{1}{2}(u^2+v^2+w^2) - \Phi + f : t$, in which the equation has the condition that the quantity q is a function of only p ; from another reason, if this equation have the condition that the value is necessary to be positive, and q is the function belonging to p,[4] then this quantity turns into $2gS - \frac{1}{2}(u^2+v^2+w^2) - \Phi$.

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(2) is the necessary condition for (1) to be the *exact differential*, and if the domain K is a simply connected domain, (2) becomes also the sufficient condition at once. We treat bellow, under the same definition about the *exact differential* and the *complete differential*.

[3]Two examples for the exact differential by Clairaut are simple : on $\frac{xdy-ydx}{x^2+y^2}$, by putting $P = -\frac{y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$ and then we get : $\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$. As well as on $\frac{ydx+xdy}{2\sqrt{a^2+xy}}$, by putting $P = \frac{y}{2\sqrt{a^2+xy}}$ and $Q = \frac{x}{2\sqrt{a^2+xy}}$ then we get : $\frac{\partial P}{\partial y} = \frac{2\sqrt{a^2+xy}-y(a^2+xy)^{-\frac{1}{2}}x}{(2\sqrt{a^2+xy})^2} = \frac{\partial Q}{\partial x}$. But as the two examples for inexact differential, on $x^2dy + y^2dx$, we get $\frac{\partial P}{\partial y} = 2y \neq \frac{\partial Q}{\partial x} = 2x$. And on $x^3dy + y^3dx$, we get $\frac{\partial P}{\partial y} = 3y^2 \neq \frac{\partial Q}{\partial x} = 3x^2$.

[4]This is called the barotropic fluid, which follows the relation : $q = f(p)$.

1.4. **Lagrange's velocity potential** φ. Lagrange cites Euler's style, however, uses first as the velocity potential : φ which is the symbol in the modern convention.

§14. nous supposerons de plus que les forces accélératrices P, Q, R du fluide soient telles, que $Pdx + Qdy + Rdz$ soit une différentielle complète ; ce qui a lieu, en général, lorsque ces forces viennent d'une ou de plusieurs sttractions propotionelles à des fonctions quelconques des distances.

De cette manière, si l'on fait $dV = Pdx + Qdy + Rdz$, la équation proposée étant divisée par Δ se réduira à cette forme

$$\Big(\frac{dp}{dt}+p\frac{dp}{dx}+q\frac{dp}{dy}+r\frac{dp}{dz}\Big)dx+\Big(\frac{dq}{dt}+p\frac{dq}{dx}+q\frac{dq}{dy}+r\frac{dq}{dz}\Big)dy+\Big(\frac{dr}{dt}+p\frac{dr}{dx}+q\frac{dr}{dy}+r\frac{dr}{dz}\Big)dz=dV-\frac{d\Pi}{\Delta}.$$

Ainsi le premier membre de cette équation devra être en particulier une différentielle complète relativement à x, y, z, puisque le second en est une.

Qu'on retranche de part et d'autre la différentielle de $\frac{p^2+q^2+r^2}{2}$ prise relativement à x, y, z, laquelle est $\left(p\frac{dp}{dx}+q\frac{dp}{dy}+r\frac{dp}{dz}\right)dx+\left(p\frac{dq}{dx}+q\frac{dq}{dy}+r\frac{dq}{dz}\right)dy+\left(p\frac{dr}{dx}+q\frac{dr}{dy}+r\frac{dr}{dz}\right)dz$; on aura, en ordonnant les termes, cette transformée

$$\begin{aligned}&\frac{dp}{dt}dx+\frac{dq}{dt}dy+\frac{dr}{dt}dz\\&+\Big(\frac{dp}{dy}-\frac{dq}{dx}\Big)(qdx-pdy)+\Big(\frac{dp}{dz}-\frac{dr}{dx}\Big)(rdx-pdz)+\Big(\frac{dq}{dz}-\frac{dr}{dy}\Big)(rdy-qdz)=dV-\frac{d\Pi}{\Delta}-\frac{p^2+q^2+r^2}{2}.\end{aligned}$$

Donc le primier membre de cette équation devra être pareillement une différentielle exacte.

§15. li est visible que, si l'on suppose que la quantité $pdx + qdy + rdz$ soit ellemême la différentielle exacte d'une fonction quelconque φ composé de x, y, z et t, on aura $p = \frac{d\varphi}{dx}$, $q = \frac{d\varphi}{dy}$, $r = \frac{d\varphi}{dz}$. Donc $\frac{dp}{dt} = \frac{d^2\varphi}{dtdx}$, $\frac{dq}{dt} = \frac{d^2\varphi}{dtdy}$, $\frac{dr}{dt} = \frac{d^2\varphi}{dtdz}$, $\frac{dp}{dy} = \frac{d^2\varphi}{dxdy}$, $\frac{dq}{dx} = \frac{d^2\varphi}{dydx}$, $,\cdots,$ Ainsi l'équation précédente deviendra par ces substitutions

$$\frac{d^2\varphi}{dtdx}dx+\frac{d^2\varphi}{dtdy}dy+\frac{d^2\varphi}{dtdz}dz=dV-\frac{d\Pi}{\Delta}-\frac{p^2+q^2+r^2}{2},$$

laquelle est évidement intégrable par raport à x, y, z ; de sorte qu'en intégrant, on aura $\frac{d\varphi}{dt} = V - \int\frac{d\Pi}{\Delta} - \frac{p^2+q^2+r^2}{2}$. [23, pp.710-711]

1.5. **Navier's equation of fluid equilibrium.**

Navier deduces the expressions of forces of the molecular action which is under the state of motion as follows : [5]

We consider the two molecules M and M'. x, y, z are the values of the rectangular coordinates of M and $x+\alpha, y+\beta, z+\gamma$ are the values of the rectangular coordinates of M'. The length of a rayon emitting from M : $\rho = \sqrt{\alpha^2+\beta^2+\gamma^2}$. The velocity of the molecule M are u, v, w and that of the molecules M' are

$$\delta x+\frac{d\delta x}{dx}\alpha+\frac{d\delta x}{dy}\beta+\frac{d\delta x}{dz}\gamma,\quad \delta y+\frac{d\delta y}{dx}\alpha+\frac{d\delta y}{dy}\beta+\frac{d\delta y}{dz}\gamma,\quad \delta z+\frac{d\delta z}{dx}\alpha+\frac{d\delta z}{dy}\beta+\frac{d\delta z}{dz}\gamma,$$

$$\text{where,}\quad \alpha=\rho\cos\psi\cos\varphi,\quad \beta=\rho\cos\psi\sin\varphi,\quad \gamma=\rho\sin\psi, \tag{3}$$

$$\delta\alpha=\frac{d\delta x}{dx}\alpha+\frac{d\delta x}{dy}\beta+\frac{d\delta x}{dz}\gamma,\quad \delta\beta=\frac{d\delta y}{dx}\alpha+\frac{d\delta y}{dy}\beta+\frac{d\delta y}{dz}\gamma,\quad \delta\gamma=\frac{d\delta z}{dx}\alpha+\frac{d\delta z}{dy}\beta+\frac{d\delta z}{dz}\gamma.$$

$$\delta\rho=\frac{\alpha\delta\alpha+\beta\delta\beta+\gamma\delta\gamma}{\rho}.$$

$$\delta\rho=\frac{1}{\rho}\Big(\frac{d\delta x}{dx}\alpha^2+\frac{d\delta x}{dy}\alpha\beta+\frac{d\delta x}{dz}\alpha\gamma+\frac{d\delta y}{dx}\alpha\beta+\frac{d\delta y}{dy}\beta^2+\frac{d\delta y}{dz}\beta\gamma+\frac{d\delta z}{dx}\alpha\gamma+\frac{d\delta z}{dy}\beta\gamma+\frac{d\delta z}{dz}\gamma^2\Big)$$

$$\text{where}\quad \frac{d\delta x}{dy}\alpha\beta+\frac{d\delta y}{dx}\alpha\beta=0,\quad \frac{d\delta y}{dz}\beta\gamma+\frac{d\delta z}{dy}\beta\gamma=0,\quad \frac{d\delta x}{dz}\alpha\gamma+\frac{d\delta z}{dx}\alpha\gamma=0.$$

[5] Navier ([29, pp.399-405])

Here, $f(\rho)$ is a function depends on the distance ρ between M and M'. We define that ψ is the angle of the rayon ρ with its projection on the $\alpha\beta$-plane and φ is the angle which this projection forms with the α axis, and then we can evaluate only the terms as follows : $\frac{8f(\rho)}{\rho}\left(\frac{d\delta x}{dx}\alpha^2 + \frac{d\delta y}{dy}\beta^2 + \frac{d\delta z}{dz}\gamma^2\right)$. Then we evaluate finally the following using the polar system (3)

$$8\int_0^\infty d\rho\rho^3 f(\rho)\int_0^{\frac{\pi}{2}} d\psi\int_0^{\frac{\pi}{2}} d\varphi\Big(\frac{d\delta x}{dx}\cos^3\psi\cos^2\varphi + \frac{d\delta y}{dy}\cos^3\psi\sin^2\varphi + \frac{d\delta z}{dz}\sin^2\psi\cos\psi\Big).$$

$$\text{Here,}\quad \int_0^{\frac{\pi}{2}} d\psi\cos^3\psi = \frac{2}{3},\quad \int_0^{\frac{\pi}{2}} d\psi\sin^2\psi\cos\psi = \frac{1}{3},\quad \int_0^{\frac{\pi}{2}} d\varphi\cos^2\varphi = \int_0^{\frac{\pi}{2}} d\varphi\sin^2\varphi = \frac{\pi}{4},$$

It turns into : $8\frac{2}{3}\frac{\pi}{4}\int_0^\infty d\rho\rho^3 f(\rho)\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right)$. Here for the brevity, $\frac{4\pi}{3}\int_0^\infty d\rho\rho^3 f(\rho) \equiv p$, where, p depends not on the distance ρ, but only on the coordinates of x, y, z which determine the situation of the molecule M. Hence we get $p\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right)$. The equation desclibing condition of equiliblium of the system is : $0 = \iiint dxdydz\left[p\left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz}\right) + P\delta x + Q\delta y + R\delta z\right]$. By the partial integration we get

$$\begin{aligned} 0 = &\iiint dxdydz\Big[\Big(P - \frac{dp}{dx}\Big)\delta x + \Big(Q - \frac{dp}{dy}\Big)\delta y + \Big(R - \frac{dp}{dz}\Big)\delta z\Big] \\ - &\iint dydz\Big(p'\delta x' - p''\delta x''\Big) - \iint dxdz\Big(p'\delta y' - p''\delta y''\Big) - \iint dxdy\Big(p'\delta z' - p''\delta z''\Big), \end{aligned}$$

1.5.1. The indeterminate equations.

Navier reduces the indeterminate equations for fluid equilibrium into two cases.

- Exact differential for the conditions of the equilibrium of the arbitrary, interior point of the fluid,

$$\frac{dp}{dx} = P,\quad \frac{dp}{dy} = Q,\quad \frac{dp}{dz} = R,\quad dp = Pdx + Qdy + Rdz,\quad p = \int(Pdx + Qdy + Rdz) + const.$$

As the result, Navier explains exact differential for the conditions of fluid equilibrium as follows :

> formule où la fonction sous le signe $\int$ doit être nécessairement susceptible d'une *intégration exacte*, pour que le fluide soumis à l'action des forces représentées par P, Q, R, puisse demeurer en équilibre. [30, p.396].

- The boundary condition to surface,
  Navier explains the mathematical method citing Lagrange[24, pp.221-223, §29-30] as follows : regarding the conditions which react at the points of the surface of the fluid, if we substitute
  - $dydz \longrightarrow ds^2\cos l$, l : the angles by which the tangent plane makes on the surface frame with the plane yz,
  - $dxdz \longrightarrow ds^2\cos m$, m : samely, the angles with the plane xz,
  - $dxdy \longrightarrow ds^2\cos n$, n : samely, the angles with the plane xy,
  - $\iint dydz, \iint dxdz, \iint dxdy \longrightarrow \mathbf{S}ds^2$

  Hence we get as follows :

$$0 = \mathbf{S}ds^2[(p'\cos l'\delta x' - p''\cos l''\delta x'') + (p'\cos m'\delta y' - p''\cos m''\delta y'') + (p'\cos n'\delta z' - p''\cos n''\delta z'')],$$

$$0 = \int(Pdx + Qdy + Rdz) + const.$$

We get the differential equation : $0 = Pdx + Qdy + Rdz$. And among the variation $\delta x, \delta y, \delta z$, we reduce the following relation : $0 = \delta x\cos l + \delta y\cos m + \delta z\cos n$.

Navier cites the molecular theory by Laplace and chooses consistently repulsive force in Navier's papers [29, 30] as the function depending on the distance between molecules, however, N.Bowditch[6] points out that Laplace rethinks the repulsion theory and changes it, in 1819 : $\varphi(f) = A(f) - R(f)$, where $\varphi(f)$: a function depending on the distance f between the moleculars, $A(f)$: attractive force, $R(f)$: repulsive force.

1.6. Helmholtz's vorticity equations.

[6] N.Bowditch[25, p.685]

1.6.1. **Helmholltz's definition of irrotation.**

Helmholtz uses Euler's equations (1_H), because it is called that he had not known until then about Navier's equations.

$$(1_H)\quad \begin{cases} X - \frac{1}{h}\frac{dp}{dx} = \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}, \\ Y - \frac{1}{h}\frac{dp}{dy} = \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}, \\ Z - \frac{1}{h}\frac{dp}{dz} = \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}, \\ \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \end{cases} \Rightarrow \begin{cases} F - \frac{1}{h}\nabla p = \frac{du}{dt} + \mathbf{u}\cdot\nabla\mathbf{u}, \\ \nabla\cdot\mathbf{u} = 0, \\ \text{where } F \equiv (X,Y,Z), \quad \mathbf{u} \equiv (u,v,w). \end{cases} \tag{4}$$

We consider not only the forces X, Y and Z of the potential V : $(1a_H)$ $X = \frac{dV}{dx}$, $Y = \frac{dV}{dy}$, $Z = \frac{dV}{dz}$, but also moreover, *Geschwindigkitespotential* φ (velocity potential), so that :

$$(1b_H)\quad u = \frac{d\varphi}{dx}, \quad v = \frac{d\varphi}{dy}, \quad w = \frac{d\varphi}{dz}. \tag{5}$$

From the consevative law of (4) (= 1_H) we get also $\Delta\varphi = 0$.

Helmholtz does not mention explicitly about *vollständigen Differentialien* (*exact differential* or *complete differential*), however from (5) we get as follows : $(1c_H)$ $\frac{du}{dy} - \frac{dv}{dx} = 0$, $\frac{dv}{dz} - \frac{dw}{dy} = 0$, $\frac{dw}{dx} - \frac{du}{dz} = 0$, $\Rightarrow \nabla\times\mathbf{u} = \mathbf{0}$. To study these three conditions ($1c_H$), Helmholtz, considering an infinitely small volume of water in a time period dt, makes investigation comprehensively into the variation from the following three various motions :

- einer Fortfüfrung des Wassertheilchens durch den Raum hin,
- einer Ausdehnung oder Zusammenziehung des Theilchend nach drei Hauptdilationsrichtungen, wobei ein jedes aus Wasser gebildete rechtwinklige Parallelepipedon, dessen Seiten den Hauptdilationsrichtungen parallel sind, rechtwinkelig bleibt, während seine Seiten zwar ihre Länge ändern, aber ihren früheren Richtungen parallel bleiben,
- einer Drehung um eine beliebig gerichtete temporäre Rotationsaxe, welche Drehung nach einem bekannten Satze immer als Resultante dreier Drehungen um die Coordinataxen angesehen werden kann.

$$\begin{cases} u \equiv A, & \frac{du}{dx} \equiv a, \\ v \equiv B, & \frac{dv}{dy} \equiv b, \\ w \equiv C, & \frac{dw}{dz} \equiv c, \end{cases} \quad \begin{cases} \frac{dw}{dy} = \frac{dv}{dz} \equiv \alpha, \\ \frac{du}{dz} = \frac{dw}{dx} \equiv \beta, \\ \frac{dv}{dx} = \frac{du}{dy} \equiv \gamma \end{cases} \quad \cdots \text{ exact differential condition}$$

When we consider now a molecule with the coordinates : x, y and z are in infinitely small distance from $\tilde{x}, \tilde{y}$ and $\tilde{z}$, then

$$\begin{cases} u = A + a(x-\tilde{x}) + \gamma(y-\tilde{y}) + \beta(z-\tilde{z}), \\ v = B + \gamma(x-\tilde{x}) + b(\tilde{y}-y) + \alpha(z-\tilde{z}), \\ w = C + \beta(x-\tilde{x}) + \alpha(y-\tilde{y}) + c(z-\tilde{z}), \end{cases} \Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \begin{bmatrix} a & \gamma & \beta \\ \gamma & -b & \alpha \\ \beta & \alpha & c \end{bmatrix} \begin{bmatrix} x-\tilde{x} \\ y-\tilde{y} \\ z-\tilde{z} \end{bmatrix} \tag{6}$$

When we put

$$\begin{aligned} \varphi = \; & A(x-\tilde{x}) + B(y-\tilde{y}) + C(z-\tilde{z}) + \frac{1}{2}a(x-\tilde{x})^2 + \frac{1}{2}b(\tilde{y}-y)^2 + \frac{1}{2}c(z-\tilde{z})^2 \\ + \; & \alpha(y-\tilde{y})(z-\tilde{z}) + \beta(x-\tilde{x})(z-\tilde{z}) + \gamma(x-\tilde{x})(y-\tilde{y}), \end{aligned}$$

then $u = \frac{d\varphi}{dx}$, $v = \frac{d\varphi}{dy}$, $w = \frac{d\varphi}{dz}$. Moreover when we choice suitable value of coordinate x_1, y_1 and z_1 at the middle point of $\tilde{x}, \tilde{y}, \tilde{z}$: $\varphi = A_1x_1 + B_1y_1 + C_1z_1 + \frac{1}{2}a_1{x_1}^2 + \frac{1}{2}b_1{y_1}^2 + \frac{1}{2}c_1{z_1}^2$, The values of velocity u_1, v_1 and w_1, desolved into these new coordinate axis are : $u_1 = A_1 + a_1x_1$, $v_1 = B_1 + b_1y_1$, $w_1 = C_1 + c_1z_1$.

1.6.2. **Helmholltz's deduction of rotation in vorticity equations. - Helmholtz's decomposition.**

Next, Helmholtz assumes the conditions of a rotatory motion as follows :

- We consider the rotatory motion of an infinitely small mass of water of the point $\tilde{x}, \tilde{y}$ and $\tilde{z}$.
- The rotation are around the axis on a pararellel with the x, y and z.
- The mass goes through the point $\tilde{x}, \tilde{y}$ and $\tilde{z}$, with the angles of the velocity are ξ, η and ζ.

We get the components of velocity which are brought about, on a pararellel with the coordinate axis x, y and z are as follows :

$$\begin{bmatrix} 0 & (z-\tilde{z})\xi & -(y-\tilde{y})\xi \\ -(z-\tilde{z})\eta & 0 & (x-\tilde{x})\eta \\ (y-\tilde{y})\zeta & -(x-\tilde{x})\zeta & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & (y-\tilde{y})\zeta & -(z-\tilde{z})\eta \\ -(x-\tilde{x})\zeta & 0 & (z-\tilde{z})\xi \\ (x-\tilde{x})\eta & -(y-\tilde{y})\xi & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} \begin{bmatrix} x-\tilde{x} \\ y-\tilde{y} \\ z-\tilde{z} \end{bmatrix} \tag{7}$$

Then we get the responce tensor compounding from (6) and (7) :

$$\begin{bmatrix} a & \gamma & \beta \\ \gamma & -b & a \\ \beta & a & c \end{bmatrix} + \begin{bmatrix} 0 & \zeta & -\eta \\ -\zeta & 0 & \xi \\ \eta & -\xi & 0 \end{bmatrix} = \begin{bmatrix} a & (\gamma+\zeta) & (\beta-\eta) \\ (\gamma-\zeta) & -b & (\alpha+\xi) \\ (\beta+\eta) & (\alpha-\xi) & c \end{bmatrix}$$

$$\begin{cases} u = A + a(x-\tilde{x}) + (\gamma+\zeta)(y-\tilde{y}) + (\beta-\eta)(z-\tilde{z}), \\ v = B + (\gamma-\zeta)(x-\tilde{x}) + b(\tilde{y}-y) + (\alpha+\xi)(z-\tilde{z}), \\ w = C + (\beta+\eta)(x-\tilde{x}) + (\alpha-\xi)(y-\tilde{y}) + c(z-\tilde{z}), \end{cases} \Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} + \begin{bmatrix} a & (\gamma+\zeta) & (\beta-\eta) \\ (\gamma-\zeta) & -b & (\alpha+\xi) \\ (\beta+\eta) & (\alpha-\xi) & c \end{bmatrix} \begin{bmatrix} x-\tilde{x} \\ y-\tilde{y} \\ z-\tilde{z} \end{bmatrix}$$

By differentiating u, v and w with respect to x, y and z respectively and then it turns out the following vorticity equations :

$$\begin{bmatrix} a & (\gamma+\zeta) & (\beta-\eta) \\ (\gamma-\zeta) & -b & (\alpha+\xi) \\ (\beta+\eta) & (\alpha-\xi) & c \end{bmatrix} \Rightarrow \quad (2_H) \quad \begin{cases} \frac{dv}{dz} - \frac{dw}{dy} = 2\xi, \\ \frac{dw}{dx} - \frac{du}{dz} = 2\eta, \\ \frac{du}{dy} - \frac{dv}{dx} = 2\zeta. \end{cases} \Rightarrow \quad \frac{1}{2}(\nabla \times \mathbf{u}) = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \equiv \mathbf{W} \tag{8}$$

1.7. Thomson's circulation theorem and the criterion of the irrotation on the complete differential.

Thomson defines the Helmholtz-like *velocity potential* as follows : Thomson's propositions which are called Thomson's circulation theorem are as follows :

Prop 1.1. *The line-integral of the tangential component velocity round any closed curve of a moving fluid remains constant through all time.* [43, p.50]

Prop 1.2. *The rate of augmentation, per unit of time, of the space integral of the velocity along any terminated arc of the fluid is equal to the excess of the value of $\frac{1}{2}q^2 - p$, at the end towards which tangential velocity is reckoned as positive, above its value at the other end.* [43, p.50]

He explains the condition of the complete differential as the criterion of the irrotation as follows : §59(e). The condition that $udx + vdy + wdz$ is a complete differential [proved above (§13) to be the criterion of irrotational motion] means simply

- *That the flow* [defined §60 (a)] *is the same in all different mutually reconcilable lines from one to another of any two points in the fluid* ; or which is the same thing,
- *That the circulation* [§60 (a)] *is zero round every closed curve capable of being contracted to a point without passing out of a portion of the fluid through which the criterion holds.* [43, p.50]

His definitions are as follows :

> §60. *Definitions and elementary propositions.*
> - (a) The line-integral of the tangental component velocity along any finite line, straight or curved, in a moving fluid, is called the *flow* in that line. If the line is endless (that is, if it forms a closed curve or polygon), the *flow* is called *circulation.* [43, p.51]

1.8. Disputes on Helmholtz's paper.

1.8.1. Bertrand's criticism on Helmholtz's definition of *rotation*.

Bertrand[1, 2, 3, 4] and Saint-Venant[38] discuss about Helmholtz's theorem. Bertrand always critisizes Helmholtz's. As the *decisive* example of the motion along the olny z-axis Bertrand says : $\xi = 0$, $\eta = 0$ and $\zeta = \frac{1}{2}$.

> Supposons, par exemple, en adoptant la notation de M. Helmholtz, ··· Les formules de M.Helmholtz nous donnent cependant, dans ce cas, $\xi = 0$, $\eta = 0$ and $\zeta = \frac{1}{2}$, et feraient croire que chaque molécule tourne uniformément autour d'un parallèle à l'axe des z.
>
> Un tel exemple n'est-il pas décisif ? [2, p.268].

1.8.2. **Helmholtz's responces to Bertrand.**
Helmholtz responses to Bertrand as follows :

> Par la méthode décomposition choisie par moi, j'ai aussi fixé, comme on voit, le sens dans lequel il faut prendre le terme *rotation* dans mon Mémoire.
>
> Nommous u, v, w le composantes de la vitesse parallèles aux axes des coordonnées x, y, z. Alors le résultat de mon analyse préliminaire, qui semble être l'object de la critique de M.Bertrand, est celui-ci. Si l'expression $(udx+vdy+wdz)$ est une différentielle exacte, il n'y a pas de rotation dans la partie du fluid correspondant. Si cette expression n'est pas une différentielle exacte, il y a rotation. [19, p.136]

2. Proofs of the eternal continuity in time and space of an exact differential

2.1. Lagrange's first proof.

At the first time, Lagrange proves the exernity of time for the *exact differential* in 1781 and uses φ as the symbol of the velocity potential.

$$\begin{cases} p = p' + p''t + p'''t^2 + \cdots, \\ q = q' + q''t + q'''t^2 + \cdots, \\ r = r' + r''t + r'''t^2 + \cdots, \end{cases} \qquad \begin{cases} \alpha = \alpha' + \alpha''t + \alpha'''t^2 + \cdots, \\ \beta = \beta' + \beta''t + \beta'''t^2 + \cdots, \\ \gamma = \gamma' + \gamma''t + \gamma'''t^2 + \cdots, \end{cases}$$

$$\text{where,} \quad \begin{cases} \frac{dp}{dy} - \frac{dq}{dx} \equiv \alpha, \\ \frac{dp}{dz} - \frac{dr}{dz} \equiv \beta, \\ \frac{dq}{dz} - \frac{dr}{dy} \equiv \gamma, \end{cases} \quad \begin{cases} \frac{dp'}{dy} - \frac{dq'}{dx} \equiv \alpha', \\ \frac{dp'}{dz} - \frac{dr'}{dz} \equiv \beta', \\ \frac{dq'}{dz} - \frac{dr'}{dy} \equiv \gamma', \end{cases} \quad \begin{cases} \frac{dp''}{dy} - \frac{dq''}{dx} \equiv \alpha'', \\ \frac{dp''}{dz} - \frac{dr''}{dz} \equiv \beta'', \\ \frac{dq''}{dz} - \frac{dr''}{dy} \equiv \gamma'', \end{cases} \quad \cdots$$

$$\frac{dp}{dt}dx + \frac{dq}{dt}dy + \frac{dr}{dt}dz + \alpha(qdx - pdy) + \beta(rdx - pdz) + \gamma(rdy - qdz)$$

Substituting the differential and order it with respect to the power of t, then it turns into :

$$\begin{aligned} & \Big[\quad (p''dx + q''dy + r''dz) \\ + & \qquad \alpha'(q'dx - p'dy) + \beta'(r'dx - p'dz) + \gamma'(r'dy - q'dz) \Big] \\ + & \; t \Big[2(p'''dx + q'''dy + r'''dz) \\ + & \qquad \alpha'(q''dx - p''dy) + \beta'(r''dx - p''dz) + \gamma'(r''dy - q''dz) \\ + & \qquad \alpha''(q'dx - p'dy) + \beta''(r'dx - p'dz) + \gamma''(r'dy - q'dz) \Big] \\ + & \; t^2 \Big[3(p^{(4)}dx + q^{(4)}dy + r^{(4)}dz) \\ + & \qquad \alpha'(q'''dx - p'''dy) + \beta'(r'''dx - p'''dz) + \gamma'(r'''dy - q'''dz) \\ + & \qquad \alpha''(q''dx - p''dy) + \beta''(r''dx - p''dz) + \gamma''(r''dy - q''dz) \\ + & \qquad \alpha'''(q'dx - p'dy) + \beta'''(r'dx - p'dz) + \gamma'''(r'dy - q'dz) \Big] \\ + & \qquad \cdots \end{aligned} \tag{9}$$

For this value become an exact differential which is independent on t, the term of t must become an exact differential. If we suppose that $p'dx + q'dy + r'dz$ be an exact differential, then $\alpha' = \beta' = \gamma' = 0$. Hence,

- the first value of (9) which must be an exact differential turns into $p''dx + q''dy + r''dz$. If we suppose that $p''dx + q''dy + r''dz$ be an exact differential, then the conditions $\alpha'' = \beta'' = \gamma'' = 0$ are necessary.
- the second value led with t of (9) which must be an exact differential will be reduced to $2(p'''dx + q'''dy + r'''dz)$, then it is necessary that $\alpha''' = \beta''' = \gamma''' = 0$.
- the third value led with t^2 of (9) which must be an exact differential will be reduced to $3(p^{(4)}dx + q^{(4)}dy + r^{(4)}dz)$, and then it is necessary that $\alpha^{(4)} = \beta^{(4)} = \gamma^{(4)} = 0$.
- $\cdots$.

Hence if we suppose that $p'dx + q'dy + r'dz$ be an exact differential,

$$p''dx + q''dy + r''dz, \quad p'''dx + q'''dy + r'''dz, \quad p^{(4)}dx + q^{(4)}dy + r^{(4)}dz \quad \cdots,$$

must be an exact differential, when the time t is supposed to be infinitesimally small.

> Il s'ensuit de là que, si la quantité : $pdx + qdy + rdz$ est une *différentielle exacte* lorsque $t = 0$, elle devra l'être aussi lorsque t aura une value quelconque très-petit ; d'ou l'on peut conclure, en génétreral, que cette quantité devra être toujours une *différentielle exacte*, quelle que soit la valeur de t. Car puisqu'elle doit l'être depuis $t = 0$ jusqu'à $t = \theta$ (θ étant une quantité quelconque donnée très-petit) , si l'on y substitue partout $\theta + t'$ à la place de t, on prouvera de même qu'elle devra être une *différentielle exacte* depuis $t' = 0$ jusqu'à $t' = \theta$ par conséquent elle le sera depuis $t = 0$ jusqu'à $t = 2\theta$; et ainsi de suite.
>
> Donc, en général, comme l'origine des t est arbitraire, et qu'on peut prendre également t positif ou négatif, il s'ensuit que si la quantité : $pdx + qdy + rdz$ est une *différentielle exacte* dans un instant quelconque, elle devra l'être pour tous les autres instants. Par conséquent, s'il y a un seul instant dans lequel elle ne soit pas une *différentielle exacte*, elle ne pourra jamais l'être pendant tout le mouvement ; car si elle l'étant dans un autre instant quelconque, elle devrait l'être aussi dans le premier. [23, §19, p.716-717].

Lagrange's claim is as follows : we suppose at first θ as the small value and t in the interval of $0 \le t \le \theta$. Next, we substitute t with $\theta + t'$, and moving t' in the interval of $0 \le t' \le \theta$ then we get $0 \le t \le 2\theta$. We substitute t likely and iteratively. At last, we get that if it satisfies the exact differential of $pdx + qdy + rdz$ at $t = 0$, then also until $0 \le^{\forall} t \le \infty$.

2.2. Cauchy's proof.

$$(1_C) \quad u_0\delta + \frac{\partial q_0}{\partial a} = 0, \quad v_0\delta + \frac{\partial q_0}{\partial b} = 0, \quad w_0\delta + \frac{\partial q_0}{\partial c} = 0. \tag{10}$$

From (10), we get : (3_C) $\frac{\partial u_0}{\partial b} = \frac{\partial v_0}{\partial a}, \quad \frac{\partial u_0}{\partial c} = \frac{\partial w_0}{\partial a}, \quad \frac{\partial v_0}{\partial c} = \frac{\partial w_0}{\partial b}$.

$$\begin{cases} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial z}{\partial c})}\left[\left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial z}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial z}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial z}{\partial a}\right], \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial y}{\partial c})}\left[\left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial y}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial y}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial y}{\partial a}\right], \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial y}{\partial c})}\left[\left(\frac{\partial u_0}{\partial b} - \frac{\partial v_0}{\partial a}\right)\frac{\partial x}{\partial c} + \left(\frac{\partial w_0}{\partial a} - \frac{\partial u_0}{\partial c}\right)\frac{\partial x}{\partial b} + \left(\frac{\partial v_0}{\partial c} - \frac{\partial w_0}{\partial b}\right)\frac{\partial x}{\partial a}\right]. \end{cases}$$

where S : the relative sign of the permutation of a, b, c. Stokes explains Cauchy's S as follows :

> S is a function of the differential coefficients of x, y and z with respect to a, b and c, which by the condition of continuity is shewn to be equal to $\frac{\rho_0}{\rho}$, ρ_0 being the initial density about the particle whose density at the time considered is ρ.

Here, we can put : $\frac{1}{S(\pm\frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\frac{\partial z}{\partial c})} = 1$.

Stokes [39] evaluates Cauchy' proof and developes his own proving with **Lemma 2.1** as follows :

> §11 $\cdots$ Since $\frac{dx}{da}$, & are finite, (for to suppose them infinite would be equivalent to supposing a discontinuity to exist in the field,) it follows at onece from the preceding equations that if $\omega_0' = 0$, $\omega_0'' = 0$, $\omega_0''' = 0$, that is if $u_0da + v_0db + w_0dc$ be an exact differential, either for the whole fluid or for any portion of it, then shall $\omega' = 0$, $\omega'' = 0$, $\omega''' = 0$, i.e. $udx + vdy + wdz$ will be an exact differential, at any subsequent time, either for the whole mass or for the above portion of it.
>
> §12 It is not from seeing the smallest flaw in M.Cauchy's proof that I propose a new one, but because it is well to view the subject in different lights, and because the proof which I am about to give does not require such long equations. $\cdots$ [39, p.108]

2.3. Stokes' proof.

Stokes proposes his new proof, prising Power[35] and criticizing Newton[31], Lagrange[23], Cauchy[5] and Poisson[33]. By the way, Stokes cites Newton's proposition XL, Theorem XIII.[31].

> Si corpus cogente vi quacunque centripeta, moveatur utcunque, & corpus aliud recta ascendat vel desendat, sintque eorum velocitates in aliquo aequlium altitudinum casu aequales, velocitates eorum in omnibus aequalibus altitudinibus erunt aequales.
>
> ⇒ If the body moving with an arbitrary centripetal force, or another bodies ascending

straightforword or decending straightforword, it take the equal velocities at any same altitude in everywhere.

Stokes says :

> I confess I cannot see that Newton in his *Principia* Lib.I, Prop. 40, has proved more than that if the velocities of the two bodies are equal increments of the distances are untimately equal : at least something additional seems required to put the proof quite out of the reach of objection.

He claims a lemma to prove that $udx + vdy + wdz$ will always remain an *exact differential* in the interval of finite time. Stokes proposes the lemma as follows :

Lemma 2.1. (Stokes) *If $\omega_1, \omega_2, \cdots, \omega_n$ are n functions of t, which satisfy the n differential equations*

$$(25_S) \quad \frac{d\omega_1}{dt} = P_1\omega_1 + Q_1\omega_2 \cdots + V_1\omega_n, \quad \cdots, \quad \frac{d\omega_n}{dt} = P_n\omega_1 + Q_n\omega_2 \cdots + V_n\omega_n,$$

where $P_1, Q_1, \cdots V_n$ may be functions of $t, \omega_1, \cdots \omega_n$, and if when $\omega_1 = 0, \omega_2 = 0, \cdots, \omega_n = 0$, none of the quantities $P_1, \cdots, V_n$ is infinite for any value of t from 0 to T, and if $\omega_1, \cdots \omega_n$ are each zero when $t = 0$, then shall each of these quantities remain zero for all values of t from 0 to T.

We suppose ρ to be a function of p and $\frac{1}{f'(p)}$, namely, here we suppose the barotropic fluid, then

$$(27_S) \quad \frac{df(p)}{dx} = X - \frac{Du}{Dt}, \quad \frac{df(p)}{dy} = Y - \frac{Dv}{Dt}, \quad \frac{df(p)}{dz} = Z - \frac{Dw}{Dt},$$

The force X, Y, Z will here be supposed to be such that $Xdx + Ydy + Zdz$ is an *exact differential*, this being the case for any forces emanating from centers, and varying as any functions of the distances. Differentiating the first equation (27_S) with respect to y, and the second with respect to x, subtracting, putting for Du/Dt and Dv/Dt their values, adding and subtracting, $du/dz.dv/dz$[7] and employing the notation of Art. 2, we obtain

$$(28_S) \quad \frac{D\omega'}{Dt} = -\Big(\frac{dv}{dw} + \frac{dy}{dz}\Big)\omega' + \frac{du}{dx}\omega'' + \frac{dv}{dx}\omega''', \quad \frac{D\omega''}{Dt} = \frac{du}{dy}\omega' - \Big(\frac{du}{dx} + \frac{dw}{dz}\Big)\omega'' + \frac{dw}{dy}\omega''',$$
$$\frac{D\omega'''}{Dt} = \frac{du}{dz}\omega' + \frac{dv}{dz}\omega'' - \Big(\frac{du}{dx} + \frac{dv}{dy}\Big)\omega'''.$$

By treating the first and third, and then the second and third of equation (27_S) in the same manner, we should obtain two more equations, $\cdots$ [39, p.111]

According to Stokes' explanation, from (27_S), we get :

$$\begin{aligned}\frac{D\omega'}{Dt} &= \frac{D}{Dt}\Big\{\frac{1}{2}\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big)\Big\} \\ &= \frac{1}{2}\Big[-\Big(\frac{dv}{dy} + \frac{dw}{dz}\Big)\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big) + \frac{dv}{dx}\frac{du}{dz} - \frac{dvdw}{dx^2} + \frac{dwdv}{dx^2} - \frac{du}{dx}\frac{dw}{dy}\Big] = -\omega' \text{ div } \mathbf{u}.\end{aligned}$$

Samely, $\frac{\omega''}{Dt} = -\omega'' \text{ div } \mathbf{u}$, $\frac{D\omega'''}{Dt} = -\omega''' \text{ div } \mathbf{u}$. Then we can arrange by the array :

$$(28_S) \quad \Rightarrow \quad \begin{bmatrix} \frac{D\omega'}{Dt} \\ \frac{D\omega''}{Dt} \\ \frac{D\omega'''}{Dt} \end{bmatrix} = \begin{bmatrix} -\Big(\frac{dv}{dy} + \frac{dw}{dz}\Big) & \frac{dv}{dx} & \frac{dw}{dx} \\ \frac{du}{dy} & -\Big(\frac{uw}{dx} + \frac{dw}{dz}\Big) & \frac{dw}{dy} \\ \frac{du}{dz} & \frac{dv}{dz} & -\Big(\frac{du}{dx} + \frac{dv}{dy}\Big) \end{bmatrix} \begin{bmatrix} \omega' \\ \omega'' \\ \omega''' \end{bmatrix} \Rightarrow \quad \frac{DW}{Dt} = -W \text{ div } \mathbf{u}, \quad (11)$$

where, $\omega' = \frac{1}{2}\Big(\frac{dw}{dy} - \frac{dv}{dz}\Big)$, $\omega'' = \frac{1}{2}\Big(\frac{du}{dz} - \frac{dw}{dx}\Big)$, $\omega''' = \frac{1}{2}\Big(\frac{dv}{dx} - \frac{du}{dy}\Big)$, $W = (\omega', \omega'', \omega''')$

> Now for points in the interior of the mass the differential coefficients $\frac{du}{dz}, \cdots$ will not be infinite, on acount of the continuity of the motion, and therefore the three equations just obtained are a particular case of equations (25_S).

Stokes concludes as follows :

[7] sic.

TABLE 2. C_1, C_2, C_3, C_4 : the constant of definitions and computing of total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name	elastic solid	moment of elastic fluid	equiliblium of fluid
1	Poisson	$C_1 = k \equiv \frac{2\pi}{15} \sum \frac{r^5}{\alpha^5} \frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3} \sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta\ g_3$ $\Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta\ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot,\cdot\}$	$C_1 = -k \equiv -\frac{1}{30\varepsilon^3} \sum r^3 \frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15} \sum \frac{r^3}{4\pi\varepsilon^3} \frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3} \sum rfr$ $= -\frac{2\pi}{3} \sum \frac{r}{4\pi\varepsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10} \sum r^3 \frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30} \sum r^3 \frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3-2)_{pf} \quad N = \frac{1}{6\varepsilon^3} \sum rfr \Rightarrow \frac{1}{6}$	$C_1 = -q \equiv \frac{1}{4\varepsilon^3} \sum \frac{r_1^2 z' R}{r}$ $C_2 = p \equiv \frac{1}{6\varepsilon^3} \sum rR$ $N = p + q\left(\frac{1}{\lambda} + \frac{1}{\lambda'}\right)$ where N : the vertical force, λ, λ' : the radii of the principal curvature
2	Navier	$C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi\ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2} \frac{\pi}{4} \frac{16}{15} = \frac{2\pi}{15}$	$C_1 = \varepsilon \equiv \frac{2\pi}{15} \int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3} \int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_4 \Rightarrow \frac{2\pi}{3}$	$C_1 = p \equiv \frac{4\pi}{3} \int_0^\infty d\rho\rho^3 f(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\psi \int_0^{\frac{\pi}{2}} d\varphi g_3$ $\Rightarrow \{\frac{2}{3}, \frac{1}{3}, \frac{\pi}{4}\} \Rightarrow \frac{8\pi}{6} = \frac{4\pi}{3}$
3	Cauchy	$C_1 = R = \frac{2\pi\Delta}{15} \int_0^\infty r^3 f(r) dr$ $= \pm\frac{2\pi\Delta}{15} \int_0^\infty [r^4 f'(r) - r^3 f(r)] dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3} \int_0^\infty r^3 f(r) dr$ $C_3 = \frac{1}{2} \int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2\alpha \cos^2\beta dp$ $= \frac{1}{2} \int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2 p \sin^2 p \sin p dp$ $= \frac{2\pi}{15}$, $C_4 = \frac{1}{2} \int_0^{2\pi} \int_0^\pi \cos^2\alpha \sin p dq dp$ $= \pi \int_0^\pi \cos^2 p \sin\ p dp = \frac{2\pi}{3}$,		
4	Saint-Venant		$C_1 = \varepsilon, \quad C_2 = \frac{\varepsilon}{3}$	
5	Stokes	$C_1 = A, \quad C_2 = B$	$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$	

> If then $udx + vdy + wdz$ is an *exact differential* for any portion of the fluid when $t = 0$, that is, if ω', ω'' and ω''' are each zero when $t = 0$, it follows from the lemma of the last article that ω', ω'' and ω''' will be zero for any value of t, and therefore $udx + vdy + wdz$ will always remain an *exact differential.* [39, p.111].

It is called that this problem is solved by Stokes' proof.

3. Formulation of the two constants theory in isotropic elasticity and Navier-Stokes equations

The partial differential equations of the elastic solid or elastic fluid are expressed by using one or the pair of C_1 and C_2 such that : in the elastic solid : $\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}$. In the elastic fluid : $\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f}$, where T_1, $T_2, \cdots$ are the tensors or terms consisting our equations. For example, in modern notation of the incompressible Navier-Stokes equations, the kinetic equation and the equation of continuity are conventionally described as follows : $\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f}, \quad \operatorname{div} \mathbf{u} = 0$, in which $-\mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p$ corresponds to $-(C_1 T_1 + C_2 T_2)$. Moreover, C_1 and C_2 are described as follows :

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 f_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 f_2 S_2, \end{cases} \quad \begin{cases} S_1 = \iint f_3 \to C_3, \\ S_2 = \iint f_4 \to C_4, \end{cases} \quad \Rightarrow \quad \begin{cases} C_1 = C_3 \mathcal{L} r_1 f_1 = \frac{2\pi}{15} \mathcal{L} r_1 f_1, \\ C_2 = C_4 \mathcal{L} r_2 f_2 = \frac{2\pi}{3} \mathcal{L} r_2 f_2. \end{cases}$$

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TABLE 3. The expression of the total moment of molecular actions by Poisson, Navier, Cauchy, Saint-Venant & Stokes

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	remark
1	Poisson [32]	elastic solid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{a^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{a^5}$		r^3		fr	
2	Poisson [33]	motion of fluid	k		$\frac{2\pi}{15}$		$\sum \frac{1}{4\pi\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$		$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{2\pi}{3}$	$\sum \frac{1}{4\pi\varepsilon^3}$		r		fr	$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
3	Poisson [33]	equiliblium of fluid	q		$\frac{1}{4}$		$\sum \frac{1}{\varepsilon^3}$	$\frac{1}{r}$		$r_i^2 z' R$		$r_i = \sqrt{x'^2 + y'^2}$
				p		$\frac{1}{6}$	$\sum \frac{1}{\varepsilon^3}$		r		R	
4	Navier [29]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$		ρ : radius
5	Navier fluid [30]	motion of fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$		ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$	
6	Navier fluid [30]	equiliblium of fluid	p		$\frac{4\pi}{3}$		$\int_0^\infty d\rho$	ρ^3		$f(\rho)$		ρ : radius
7	Saint-Venant [37]	fluid	ε	$\frac{\varepsilon}{3}$								
8	Stokes [39]	fluid	μ	$\frac{\mu}{3}$								
9	Stokes [39]	elastic solid	A	B								

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